Integrable systems, spectral curves and geometry

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Symplectic manifolds and Hamiltonians

(M^{2n}, ω) - ω - a closed nondegenerate 2-form.

Example

1. Cotangent bundle T^*X. It has a tautological 1-form σ: if $\phi \in T^*X$ and $v \in T_\phi T^*X$, then

$$\sigma_{|\phi}(v) = \langle \phi, d\pi(v) \rangle \quad (\pi : T^*X \to X).$$

The symplectic form is then $\omega = d\sigma$.

If $q_1, \dots, q_n$ are local coordinates on X and $p_1, \dots, p_n$ the induced coordinates on the fibres (in the local frame $dq_1, \dots, dq_n$), then

$$\sigma = \sum_{i=1}^{n} p_i dq_i \;\text{ and }\; \omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

2. Coadjoint orbits. G-Lie group, $\mathfrak{g}$ its Lie algebra, O a coadjoint G-orbit ($O \subset \mathfrak{g}^*$). At $X \in O$, any tangent vector is of the form $X([\rho, \cdot])$ for a $\rho \in \mathfrak{g}$.

The Kostant-Kirillov-Souriau symplectic form on O is defined by

$$\omega_{|X}\big(X([\rho_1, \cdot]), X([\rho_2, \cdot])\big) = X([\rho_1, \rho_2]).$$

To a smooth function $H : M \longrightarrow \mathbb{R}$ on a symplectic manifold we associate its *Hamiltonian vector field* X_H, which is simply the symplectic gradient:

$$dH(v) = \omega(X_H, v) \ \ \forall v \in TM,$$

and consider the corresponding vector field flow: $\dot{x}(t) = X_H$. This is called a Hamiltonian flow, and H is called a Hamiltonian. If $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$, then

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}.$$

Example Take $M = T^*\mathbb{R} \simeq \mathbb{R} \times \mathbb{R}$ with $\omega = dp \wedge dq$ and the Hamiltonian $H = p^2/2 + U(q)$ (U- a *potential*). Then $X_H = \left(p, -\frac{\partial U}{\partial q}\right)$, so that the flow is defined by the equations $\dot{q} = p$, $\dot{p} = -\frac{\partial U}{\partial q}$. These equations reduce to Newton's equation $\ddot{q} = -\frac{\partial U}{\partial q}$.

Completely integrable Hamiltonian systems

Observe that the Hamiltonian is always constant along the flow: $\frac{d}{dt}H(x(t)) = dH(\dot{x}(t)) = dH(X_H) = \omega(X_H, X_H) = 0$.
More generally, we can consider two (or more) Hamiltonians H_1, H_2 such that H_1 stays constant along the flow of H_2 and vice versa. It is easy to see that this condition is equivalent to $\omega(X_{H_1}, X_{H_2}) = 0$. One says that H_1 and H_2 *(Poisson) commute*.
Thus, if we have k commuting Hamiltonians $H_1, \ldots, H_k$, then the resulting $\mathbb{R}^k$-action leaves invariant $H_1, \ldots, H_k$. In particular, if the action is to have k-dimensional orbits, then $k \leq n$, where $\dim M = 2n$. If $k = n$ and the orbits are generically n-dimensional, then we say that our Hamiltonian system is *completely integrable* (in the sense of Liouville).
We can then take special local coordinates $q_i(x) = H_i(x)$ (action variables) and $(p_1, \ldots, p_n)(x) = t \in \mathbb{R}^n$, such that $t.x_0 = x$ (angle variables). The symplectic form becomes $\sum dp_i \wedge dq_i$ and the Hamiltonian flow for each H_i is now linear:
$t.(q_1, \ldots, q_n, p_1, \ldots, p_n) = (q_1, \ldots, q_n, p_1, \ldots, p_i - t, \ldots, p_n)$.

Remark: Locally we can always find n commuting Hamiltonians, but if $H_1, \ldots, H_n$ are defined globally, then there will be points where the differentials $dH_1, \ldots, dH_n$ are linearly dependent. These correspond to lower-dimensional (singular) orbits of the $\mathbb{R}^n$-action, and it is these orbits which determine the topology of M, its global geometry and interesting phenomena of the mechanical system.

Example: Calogero-Moser system: $H = \sum_{i=1}^n p_i^2 - \sum_{i\neq j} \frac{1}{(x_i - x_j)^2}$. It describes the motion of n particles on the line with interaction potential $-1/x^2$. Why should it be completely integrable? The answer is provided by the method of *symplectic reduction*.

Let (M, ω) be a symplectic manifold with an action of a Lie group G preserving ω. A *moment map* for G is an equivariant map $\mu : M \to \mathfrak{g}^*$ such that for any $\rho \in \mathfrak{g}$ and $v \in TM$

$$\langle d\mu(v), \rho \rangle = \omega(X_\rho, v),$$

where X_ρ is the vector field on M induced by ρ. In this situation, for any $c \in (\mathfrak{g}^*)^G$, if G acts freely (and properly) on $\mu^{-1}(c)$, then $\mu^{-1}(c)/G$ is a symplectic manifold of dimension $\dim M - 2\dim G$.

More generally, for any coadjoint orbit O, if G acts freely and properly on $\mu^{-1}(O)$, then $\mu^{-1}(O)/G$ is a symplectic manifold of dimension $\dim M - 2\dim G + \dim O$.

Example. Let $M = T^*\mathrm{Mat}_n(\mathbb{R}) \simeq \mathrm{Mat}_n(\mathbb{R}) \oplus \mathrm{Mat}_n(\mathbb{R})$ with the symplectic form $\omega = \operatorname{tr} dX \wedge dY$ (tr is used to identify $\mathrm{Mat}_n(\mathbb{R})^*$ with $\mathrm{Mat}_n(\mathbb{R})$). The group $G = PGL_n(\mathbb{R})$ acting by conjugation on both factors preserves ω and the moment map is $\mu(X, Y) = [X, Y]$. Let O be the adjoint orbit of $\operatorname{diag}(-1, \ldots, -1, n-1)$. Then $\mu^{-1}(O)/G$ is a symplectic manifold of dimension $2n^2 - 2(n^2 - 1) + 2(n-1) = 2n$. On M we have the Hamiltonians $H_i = \operatorname{tr} Y^i$, $i = 1, \ldots, n$. They clearly commute, but they do not form an integrable system, since there are too few of them. However they are also G-invariant, so they descend to $\mu^{-1}(O)/G$ and define there a completely integrable system.
Observe that $T \in O$ iff $T + I$ has rank 1. Consider the subset of $\mu^{-1}(O)/G$ with X having distinct real eigenvalues, so that X is represented by $\operatorname{diag}(x_1, \ldots, x_n)$ with $x_i \neq x_j$. Then the diagonal entries of $[X, Y]$ are zero, and since $[X, Y] + I$ has rank 1, it is of the form $[a_i a_j^{-1}]$ for some nonzero numbers a_i. We can use the remaining freedom of conjugating by diagonal matrices to make all $a_i = 1$.

Then

$$[X,Y]+I=[1] \iff Y_{ij}=\frac{1}{x_i-x_j},\ \ i\neq j.$$

The diagonal entries of Y are unconstrained; let us write $Y_{ii}=p_i$. Thus the open subset of $\mu^{-1}(O)/G$ with X having distinct real eigenvalues can be identified with $\{(x_i)\in\mathbb{R}^n; x_i\neq x_j\}\times\mathbb{R}^n$. The Hamiltonian $H_2=\operatorname{tr}Y^2$ becomes

$$\sum p_i^2+\sum_{i\neq j}\frac{1}{(x_i-x_j)}\frac{1}{(x_j-x_i)}=\sum_{i=1}^n p_i^2-\sum_{i\neq j}\frac{1}{(x_i-x_j)^2}.$$

Thus the Calogero-Moser system is completely integrable and, moreover, the argument shows that the proper completion of the phase space of this system is (the "hyperbolic" part of) $\mu^{-1}(O)/G$.

Example (hyperbolic Calogero-Moser). Let M,G,O be as above, but set $H_i=\operatorname{tr}(XY)^i$. On the same subset of $\mu^{-1}(O)$ as above we get $H_2=\sum_{i=1}^n(x_ip_i)^2-\sum_{i\neq j}\frac{x_ix_j}{(x_i-x_j)^2}$. If all $x_i>0$, we can use the coordinates $\tilde{p}_i=x_ip_i$ and $\tilde{x}_i=\log x_i$ and obtain:

$$H_2=\sum_{i=1}^n\tilde{p}_i^2-\sum_{i\neq j}\frac{4}{\sinh^2(\tilde{x}_i/2-\tilde{x}_j/2)}.$$

Lax pairs

Look again at the rational Calogero-Moser system with n commuting Hamiltonians $H_i=\operatorname{tr}Y^i$. These Hamiltonians are constants of motions, and so on the subset where Y has distinct eigenvalues, Y moves in a fixed conjugacy class: $Y(t)=g(t)Y(0)g(t)^{-1}$. Differentiating we get the equation $\dot{Y}(t)=\left[\dot{g}(t)g(t)^{-1},Y(t)\right]$, called the *Lax equation*. Conversely, if L,M are $n\times n$-matrices satisfying the equation

$$\dot{L}=[M,L],$$

then L moves in a fixed adjoint orbit, and the functions $\operatorname{tr}L^i$, $i=1,\dots,n$ are constants of motion. A pair (L,M) is called a *Lax pair*, and finding such a pair is usually a first step in showing that a system is completely integrable (and in finding the explicit solutions). For the Calogero-Moser system we have

$$L_{ij}=\delta_{ij}p_i+(1-\delta_{ij})\frac{1}{x_i-x_j},$$
$$M_{ij}=-\delta_{ij}\sum_{k\neq i}\frac{1}{(x_i-x_k)^2}+(1-\delta_{ij})\frac{1}{(x_i-x_j)^2}.$$

Finding Lax pairs for a given integrable system is highly nontrivial. Moreover, the above description, i.e. the integrable system arises from a flow on a (co)-adjoint orbit of a finite-dimensional Lie group is usually insufficient. Consider, for example, the symplectic reduction of $M = T^*\mathrm{Mat}_n(\mathbb{R}) \simeq \mathrm{Mat}_n(\mathbb{R}) \oplus \mathrm{Mat}_n(\mathbb{R})$ with respect to a non-minimal orbit $O \in SL(n,\mathbb{R})$. The dimension of the symplectic quotient is $\dim O + 2$, so that $\operatorname{tr} Y^i$, $i = 1,\dots,n$, no longer provide enough commuting Hamiltonians. Nevertheless the Hamiltonian $\operatorname{tr} Y^2$ still defines a completely integrable system *spin Calogero-Mosero system*. The solution is to consider Lax pairs with a parameter $\lambda \in \mathbb{C}$, i.e. $L(\lambda)$ and $M(\lambda)$ are rational expressions in λ. The Lax equation takes the form

$$\frac{\partial}{\partial t} L(t,\lambda) = \left[M(t,\lambda), L(t,\lambda)\right].$$

We can interpret L and M as taking values in the dual of a loop algebra, and the flow lives on a finite-dimensional coadjoint orbit of the loop group. The characteristic polynomial of L remains constant in t and it defines a plane algebraic curve S with the equation:

$$\det\left(z - L(\lambda)\right) = 0.$$

Either S itself or its smooth projective model is called the *spectral curve* of the problem.

Let us consider (for simplicity!) the case, when $L(\lambda)$ is a polynomial in λ, say of degree d. Then it is natural to assume that z lives in the total space of $\mathcal{O}(d)$ - the d-th power of the hyperplane line bundle over $\mathbb{C}P^1$ (no singularities over $\lambda = \infty$). Thus $\det\left(z - L(\lambda)\right) = 0$ is compactified in $|\mathcal{O}(d)|$ (or in the Hirzebruch surface F_d). The genus of S is $(n-1)(dn-2)/2$.

The coefficients of the equation define the commuting Hamiltonians.

What are the angle coordinates?

Theorem (Beauville)

Let S be a smooth compact curve in $|\mathcal{O}(d)|$ of genus $g = (n-1)(dn-2)/2$. There is a natural bijection between $GL(n,\mathbb{C})$-conjugacy classes of $\mathrm{Mat}_n(\mathbb{C})$-valued polynomials $L(\lambda)$ of degree d, the characteristic polynomial of which defines S, and line bundles of degree $g-1$ on S which admit no nonzero sections.

It is actually a biholomorphism between $\{L(\lambda)\}/GL(n,\mathbb{C})$ and $\mathrm{Jac}^{g-1}(S) - \Theta$ (open subset of a g-dimensional torus).

The correspondence is seen from the exact sequence (here $\mathbb{T}$ denotes the total space of $\mathcal{O}(d)$ and line bundles are pull-backs from $\mathbb{CP}^1$):

$$0 \to \mathcal{O}_{\mathbb{T}}(-d-1)^{\oplus n} \to \mathcal{O}_{\mathbb{T}}(-1)^{\oplus n} \to E \to 0,$$

where the first map is given by $z \cdot 1 - L(\lambda)$ and E is viewed as a sheaf on $\mathbb{T}$ supported on S.
In many cases the flow corresponding to the Lax equation becomes *linear* on $\mathrm{Jac}^{g-1}(S) - \Theta$, so the Jacobian really gives us the angle coordinates. One such condition is that $M(\lambda) = p(L(\lambda), \lambda)_+$, where $p(z, \lambda)$ is rational in λ and polynomial in z (and constant in t) and $+$ denotes the part polynomial in λ.
Many integrable systems (e.g. spin Calogero-Moser) fit into this framework, i.e. they can be linearised on the Jacobian of an algebraic curve. Such systems can the be solved (in principle!) explicitly in terms of the theta functions of the spectral curve.
There are also many geometric problems which lead to and can be solved via such integrable systems. I'll describe one such: construction of hyperkähler metrics via reduction of self-duality equations.

We consider curves in $\mathbb{T} = |\mathcal{O}(2)|$ and the flow on the Jacobian in the direction of $[z/\lambda] \in H^1(S, \mathcal{O})$. In other words, the flow of line bundles has the form $E_t = E_0 \otimes F^t$, where the transition function for F^t is $e^{tz/\lambda}$. From E_t, we obtain a family of vector spaces $V_t = H^0(S, E_t(1))$ and, a family of endomorphisms $\tilde{L}(t, \lambda)$. For a given t, a choice of basis of V_t produces a (quadratic in λ) matrix polynomial $L(t, \lambda)$, but without some canonical choice of the bases, there is no hope that $L(t, \lambda)$ satisfy the Lax equation.
Such a canonical choice is equivalent to saying that we chose parallel sections of the vector bundle V over $\mathbb{R}$, the fibre of which over t is V_t, i.e. to a choice of a connection on the bundle of V. One such choice is the connection ∇_0 evaluating the values (in our chosen trivialisation) of sections at points of S over a fixed λ, say $\lambda = 0$.
The resulting equations for $L(t, \lambda) = L_0(t) + L_1(t)\lambda + L_2(t)\lambda^2$ are:

$$\frac{\partial}{\partial t} L(t, \lambda) = [L_2(t)\lambda, L(t, \lambda)] .$$

We take instead the *Hitchin connection* $\nabla_0 + \frac{1}{2} L_1(t)$.

We get

$$\frac{\partial}{\partial t}L(t,\lambda) = \left[\frac{1}{2}L_1(t) + L_2(t)\lambda, L(t,\lambda)\right], \text{ i.e.}$$

$$\dot{L}_0 = \frac{1}{2}[L_1, L_0],$$

$$\dot{L}_1 = [L_2, L_0],$$

$$\dot{L}_2 = \frac{1}{2}[L_2, L_1].$$

We now assume that $L(\lambda)$ is invariant under the involution: $L(\lambda) \to -\bar{\lambda}^2 L(-1/\bar{\lambda})^*$, i.e. $L_2 = -L_0^*$, $L_1^* = L_1$. This implies in particular that the curve S is invariant under the involution $(\lambda, z) \to (-1/\bar{\lambda}, -\bar{z}/\bar{\lambda}^2)$ and that the corresponding line bundle lives on the "real" part of the Jacobian.
Setting $L_0 = T_2 + iT_3$, $L_1 = 2iT_1$ with $T_i \in \mathfrak{u}(n)$ we obtain the Nahm's equations:

$$\dot{T}_1 = [T_2, T_3], \quad \dot{T}_2 = [T_3, T_1], \quad \dot{T}_3 = [T_1, T_2].$$

We can now consider sets of solutions with different boundary conditions: solutions regular on an interval, solutions on a half-line, solutions with prescribed simple poles at one endpoint, solutions on several adjoining intervals with matching conditions at the endpoints, and also solution with values in a subalgebra of $\mathfrak{u}(n)$.
All of these sets of solutions are hyperkähler manifolds

Examples: 1. Solutions on $(0,1)$ with simple poles at 0 and 1, the residues of which define the standard n-dimensional irreducible representation of $\mathfrak{su}(2)$ describe the moduli space of charge n $SU(2)$-monopoles on $\mathbb{R}^3$ with its natural (and important!) metric. (Nahm, Hurtubise, Nakajima).
2. Smooth solutions on $[0,1]$ define a hyperkähler metric on $T^*GL(n,\mathbb{C})$. (Kronheimer)
3. Smooth solutions on $[0,+\infty)$ such that the limits belong to fixed regular conjugacy classes define hyperkähler metrics on adjoint orbits of $GL_n(\mathbb{C})$. (Kronheimer)

This last construction is particularly interesting: the hyperkähler metric is algebraic, but there is no known Lie-group-theoretic description of it (unlike, say, Kähler metrics on adjoint orbits of compact groups).
Using the above methods of integrable systems and ideas of Nigel Hitchin for monopole moduli spaces one can prove a formula for a Kähler potential of this metric, i.e. a function K such that $\omega = i\partial\bar{\partial}K$.
The spectral curve in this case is singular: a union of n copies of $\mathbb{C}P^1$, each pair having two intersection points. The Jacobian $\mathrm{Jac}^{g-1}(S)$ is isomorphic to $(\mathbb{C}^*)^g$ and the theta function ϑ is a polynomial (similar to a case considered by Mumford).
Then

$$K = X(\log\vartheta),$$

where X is the vector field on $\mathrm{Jac}^{g-1}(S)$ generated by the action of $[\exp(tz/\lambda)]$.